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Representation-Functors and Flag-algebras for the Classical Groups I

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In [8] and [9], one of the present authors constructed new functors from modules (over any communtative ring) to algebras; this work utilized a new basis-free construction for the finite-dimensional irreducible polynomial representations of $GL(n)$. These results extended prior constructions by Higman [5], Epstein [3], Beetham [1], and Carter and Lusztig [2]; for a fuller discussion of the relation of the functors in [8] with these earlier constructions, as well as with constructions which have since appeared (e.g., in [6]), the reader is referred to [9].

In the present paper and its sequel [11], analogous "basis-free" constructions will be given for the other classical groups; that is, suitable functors will be constructed from the category of finitely generated projective symplectic (or "oriented quadratic") modules to that of finitely generated modules, related to the representation theory of B_n, C_n, D_n as the previously mentioned functors are related to the representation theory of GL_n; such functors will also be constructed for the representation theory of the orthogonal group. (One drawback: these constructions do not catch the spin representations).

One possible area of application of these functors will be indicated in [11], where they are used to extend to the classical groups, some constructions (for the special linear group) of 19th century invariant theory.

The construction of these representation functors, rests on a study of the ideal of the isotropic flag variety for the groups in question, and requires obtaining a generating set for this ideal; this continues the program begun by one of the authors in [9], and there carried out in some detail for the general linear group.

The material in [8] and [9] has been extended by recent work of C. Procesi and his school; in particular, De Concini has obtained some important results in these directions, on the symplectic groups, utilizing recent joint work of Seshadri, Musili, and Lakshmi Bai (see Remark 3.4 below).

A theorem due to Kostant, enables one to obtain the generators for the ideals in question by a straight-forward computation. This theorem (and its proof, also

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due entirely to Kostant) are given below as Theorem 1.1; it forms an essential basis for the present work. We also wish to acknowledge our gratitude to Verma for directing our attention to this result of Kostant, and to Kostant for being kind enough to explain this matter to us in some detail.

The major new theorems obtained here, are as follows; they concern the coordinate ring, here denoted by $\Lambda^+(G)$, of the flag variety G/U of a reductive group G. The main goal is, to obtain generators and relations for the "flag-algebra" for $\Lambda^+(G)$ in a *basis-free fashion* (so the constructions apply *without modification* in the context of a projective module with symplectic or quadratic structure; the graded pieces of the resulting algebra then yield the representation-functors described above. Basis-free here also refers to the requirement that these constructions not involve a particular choice of Borel subgroup and unipotent radical U).

Let V be an n-dimensional complex vector space; generators and relations for

$$\Lambda^+ V = \Lambda^+(\operatorname{Aut} V)$$

have been described in [9]. If ϵ is an "orientation" of V, i.e. a free generator for $\Lambda^n V$, then the automorphism group of the structure (V, ϵ) is isomorphic to $SL_n(\mathbb{C})$, and it is trivial to verify that there is a natural isomorphism

$$\Lambda^+(\operatorname{Aut}(V, \epsilon)) \approx \Lambda^+ V/(\epsilon - 1) \tag{0.1}$$

A rather deeper result concerns the case when $n = 2l + 1$ is odd and V is also furnished with a non-singular symmetric bilinear form $\langle , \rangle$, with respect to which ϵ has length 1; here the automorphism group of the structure $(V, \langle , \rangle, \epsilon)$ is isomorphic to $SO(n)$, and we have the following analog of 1): there is a natural isomorphism

$$\Lambda^+(\operatorname{Aut}(V, \langle , \rangle, \epsilon)) \approx \Lambda^+ V/I_1 \tag{0.2}$$

where I_1 is the ideal in $\Lambda^+ V$ generated by all

$$^*\alpha - \alpha \qquad (\alpha \in \Lambda^p V, 0 \leqslant p \leqslant l)$$

(* denoting the Hodge star).

An analogous result holds if we assume that $n = 2l$ is even, and that $\langle \; \rangle$ is symplectic; namely, there is then a natural isomorphism

$$\Lambda^+(\operatorname{Aut}(V, \langle \; \rangle) \approx \Lambda^+ V/I_2 \tag{0.3}$$

where I_2 is the ideal in $\Lambda^+ V$ generated by all $\alpha \wedge \omega$ ($\alpha \in \Lambda^i V$ with $i \leqslant l - 2$, ω = 2-form associated to $\langle \; \rangle$), and all $^*\alpha - \alpha$ ($\alpha \in \Lambda^p V$, $0 \leqslant p < l$).

The analogous results for the even-dimensional special orthogonal group are rather more complicated. On the other hand, a single simple construction works for both the odd- and even-dimensional orthogonal groups: If $\dim V = n$, $l = [n/2]$ and $\langle \; \rangle$ is a non-singular symmetric bilinear form on V and if ϵ is an orientation of V, then $\mathrm{Aut}(V, \langle\,\rangle)$ is isomorphic to $O(n)$, and there is a natural isomorphism

$$\Lambda^+(\mathrm{Aut}(V, \langle\,\rangle) \approx (\Lambda^+V)[T]/I_3 \tag{0.4}$$

where T is an indeterminate adjoined to Λ^+V, and I_3 is the ideal in $\Lambda^+V[T]$ generated by $T^2 - \langle\epsilon, \epsilon\rangle^{-1}$ and by all $T\alpha - *_\epsilon\alpha$ ($\alpha \in \Lambda^i E$, $0 \leqslant i \leqslant l$). (More intrinsically, we may simultaneously adjoin one such T_ϵ for each orientation ϵ with the additional relations $T_{\epsilon_1}T_{\epsilon_2} = \langle\epsilon_1, \epsilon_2\rangle^{-1}$; the result is the same.)

A curious variation on (2) and (3) may also be obtained, utilizing the "extra" structure on Λ^+V described on p. 101 of [8]; this extra structure is a graded anti-commutative "wedge product" operation defined on Λ^+V. Λ^+V is thus furnished simultaneously with two multiplications, a "dot" product (the usual commutative multiplication in the coordinate ring Λ^+V of the flag variety) and also the wedge product. If we assume V is furnished with a non-singular symplectic product $\langle,\rangle$, associated with the 2-form $\omega \in \Lambda^2V$, and if we denote by $((\omega)$ the "double ideal" in Λ^+V generated by ω with respect to these two operations (i.e. the set of all sums of expressions built up using these two products, at least one term in each expression being ω) then it is proved that there is a natural isomorphism

$$\Lambda^+(\mathrm{Aut}(V, \langle\,\rangle) \approx \Lambda^+V/((\omega)) \tag{0.5}$$

(Note: this is not the same formula as 0.3), i.e. $((\omega))$ does not coincide with I_2).

There is an analogous (but less elegant) result for the odd-dimensional special orthogonal group; cf. Th. 2.4) below.

Some results of Malcev [7] indicate that the functors constructed here from the categories $\mathscr{C}_1$, $\mathscr{C}_2$ of orientable projective R-modules with non-singular quadratic resp. symplectic forms into the category of R-modules, in fact take their values in $\mathscr{C}_1$ or $\mathscr{C}_2$; the authors hope to be able soon to publish a continuation of these constructions in this direction.

The present paper only treats the cases B_l and C_l; the proof of the other results announced in this introduction will be forthcoming in [11].

1. The Coordinate-Ring of G/U

Let G be a reductive Lie group, over an algebraically closed field k of characteristic 0. We suppose chosen a Borel subgroup with unipotent radical U, and a specific ordering of the roots. We then denote the coordinate ring over k of G/U by $\Lambda^+(G)$, the "flag-algebra" of G. Note that $\Lambda^+(G)$ is a k-algebra, graded by the monoid $W(G)$ of all highest weights α of finite-dimensional irreducible polynomial representations π_α of G:

$$\Lambda^+(G) = \bigoplus_{\alpha \in W(G)} \Lambda^\alpha(G)$$

where $\Lambda^\alpha = \Lambda^\alpha(G)$ is a representation-module for π_α. Note also that the multiplication in $\Lambda^+(G)$ is the Cartan product

$$\Lambda^{\alpha_1} \otimes \Lambda^{\alpha_2} \to \Lambda^{\alpha_1+\alpha_2}$$

THEOREM 1.1 (Kostant). *Let G be semi-simple, with Killing form B, and let $\{x_1,..., x_n\}$, $\{x_1^*,..., x_n^*\}$ be dual bases with respect to B, for the Lie algebra of G; assume also that the monoid $W(G)$ is free on a finite subset W_0 (i.e., every β in $W(G)$ may be uniquely expressed in the form*

$$\beta = \sum_{\alpha \in W_0} n_\alpha \alpha$$

with each n_α a nonnegative integer). Then $\Lambda^+(G)$ is generated as k-algebra by the set

$$\bigcup_{\alpha \in W_0} \Lambda^\alpha(G) \tag{1.1}$$

with generating relations given by the union of the following sets (as α_1, α_2 run through all weights in W_0):

$$\begin{aligned}&\operatorname{Ker}(\Lambda^{\alpha_1}(G) \otimes \Lambda^{\alpha_2}(G) \to \Lambda^{\alpha_1+\alpha_2}(G))\\ &\quad = \left(\left[\sum_{i=1}^n (x_i \otimes x_i^* + x_i^* \otimes x_i)\right] - 2B(\alpha_1, \alpha_2)\, Id\right) \cdot \Lambda^{\alpha_1}(G) \otimes \Lambda^{\alpha_2}(G)\end{aligned} \tag{1.2}$$

Proof. Denote by p, half the sum of the positive roots of G, and recall that the Casimir operator

$$\omega = \sum_{i=1}^n x_i x_i^*$$

acts on each Λ^α as the scalar $B(\alpha, \alpha) + 2B(p, \alpha)$, which we denote by $\alpha[\omega]$. Since also $B(p, \lambda) > 0$, $B(\alpha, \lambda) \geqslant 0$ for any positive root α, it follows that given a second representation $\Lambda^{\alpha'}$ with $\alpha > \alpha'$, we then have $\alpha[\omega] > \alpha'[\omega]$.

Hence, if α_1,..., α_n are in W_0, the kernel K of the Cartan product

$$\Lambda^{\alpha_1} \otimes \cdots \otimes \Lambda^{\alpha_m} \to \Lambda^{\alpha_1+\cdots+\alpha_m}$$

is the image of the projection

$$\begin{aligned}P &= (\pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_m})(\omega) - (\alpha_1 + \cdots + \alpha_m)[\omega]\, Id\\ &= T_1 + \sum_{1 \leqslant r < s \leqslant n} T_{rs} - (\alpha_1 + \cdots + \alpha_m)[\omega]\, Id\end{aligned} \tag{1.3}$$

where

$$\begin{aligned}T_1(v_1 \otimes \cdots \otimes v_m) &= \sum_{i=1}^m v_1 \otimes \cdots \otimes \omega v_i \otimes \cdots \otimes v_m\\ &= \left(\sum_{i=1}^m \alpha_i[\omega]\right) (v_1 \otimes \cdots \otimes v_m)\end{aligned}$$

and

$$T_{rs} = \sum_{i=1}^{m} (v_1 \otimes \cdots \otimes x_i v_r \otimes \cdots \otimes x_i^* v_s \otimes \cdots \otimes v_m$$
$$+ v_1 \otimes \cdots \otimes x_i^* v_r \otimes \cdots \otimes x_i v_s \otimes \cdots)$$

A simple computation using (1.3) now shows

$$P(v_1 \otimes \cdots \otimes v_m) = \sum_{1 \leqslant r < s \leqslant n} (T_{rs} - 2B(\alpha_r, \alpha_s)(v_1 \otimes \cdots \otimes v_m)$$

whence the truth of the theorem is immediate.

COROLLARY. *If, in the preceding theorem, we assume only that W_0 generates $W(G)$ (not necessarily freely), it remains true that the "homogeneous" relations on* (1.1), *i.e., those in*

$$\mathrm{Ker}(\Lambda^{\alpha_1} \otimes \cdots \otimes \Lambda^{\alpha_m} \to \Lambda^{\alpha_1 + \cdots + \alpha_m})$$

are generated by the relations (1.2).

Remark. The hypotheses of Th. 1.1 are satisfied by $SL(n)$, $SO(2l + 1)$ $Sp(l)$ but not by $SO(2l)$: It is mainly for this reason that the representation theory of $SO(2l)$ (at least in the terms studied in the present paper) presents certain peculiarities (cf. Sects. 5 and 6).

2. THE ODD-DIMENSIONAL SPECIAL ORTHOGONAL GROUPS

In this section, we shall begin by some general considerations on special orthogonal groups, and then concentrate on the groups B_l, i.e., the $(2l + 1)$-dimensional special orthogonal groups.

Rather than considering the special orthogonal groups primarily as matrix groups $SO(n)$, we shall consider them as automorphism groups

$$\mathrm{Aut}(V, \langle,\rangle, \epsilon) \tag{2.1}$$

of structures consisting of an n-dimensional vector-space V, furnished with a symmetric inner product $\langle,\rangle$ and an orientation $\epsilon \in \Lambda^n V$. A general guiding principle in what follows, is that our constructions should not involve the choice of a specific basis for V, or of a specific Borel sub-group for 2.1); naturally no such restriction need apply in our proofs. Working in a more general context makes it easier to follow this principle.

For the remainder of this paper, R will denote a commutative associative ring with 1.

DEFINITION 2.1. By a *non-singular orientable quadratic R-module of rank n* will be meant an ordered pair $(P, \langle,\rangle)$ where P is an R-module and

$$\langle,\rangle: P \otimes_R P \to R$$

is a symmetric R-bilinear form on P, these being subject to the two conditions:

(i) $\Lambda^n P$ is free over R on one generator

(ii) The R-homomorphism

$$i_{\langle\,\rangle}: P \to P^* = \mathrm{Hom}_R(P, R), p \to \langle p, -\rangle$$

is an isomorphism.

These form a category, to be denoted by

$$\underline{\underline{O(n, R)}}$$

whose morphisms are the R-isomorphisms which preserve $\langle\,\rangle$.

If $(P, \langle\,\rangle)$ is in this category, there are induced non-singular symmetric bilinear forms on $\Lambda^i P$ $(1 \leqslant i \leqslant n)$ and on P^*, which (by an abuse of notation) will again be denoted by $\langle\,\rangle$. By an *orientation* for $(P, \langle\,\rangle)$ will be meant a generator ϵ for $\Lambda^n E$ over R; the triple $(P, \langle,\rangle, \epsilon)$ will then be called a *non-singular oriented quadratic R-module of rank n*; these form a category, to be denoted by

$$\underline{\underline{SO(n, R)}}$$

whose morphisms are the R-isomorphisms which preserve $\langle\,\rangle$ and ϵ.

Given $(P, \langle\,\rangle, \epsilon)$ in the latter category, the R-isomorphism

$$* = *_\epsilon = *_{\epsilon,i} : \Lambda^i P \xrightarrow{\approx} \Lambda^{n-i} P$$

(the "Hodge star") is uniquely defined for $0 \leqslant i \leqslant n$ by the requirement:

$$\omega \wedge \omega' = \langle *_\epsilon \omega, \omega' \rangle \epsilon \qquad (\text{for all } \omega \in \Lambda^i P,\ \omega' \in \Lambda^{n-i} P)$$

If $(P, \langle\,\rangle)$ is an object in $\underline{\underline{O(n, R)}}$, and if P is R-free on $\{e_1, \ldots, e_n\}$, with dual basis $\{e_1^*, \ldots, e_n^*\}$, then we set

$$\theta = \sum_{i<j} \langle e_i^*, e_j^* \rangle\, e_i e_j + \sum_i \langle e_i^*, e_i^* \rangle e_i^{(2)} \in D_2 P \tag{2.2}$$

This homogeneous element of order 2 in the divided-powers algebra of P is independent of choice of basis $\{e_1, \ldots, e_n\}$; in the general case, when P need not be free, θ is uniquely specified by the requirement that it be given by 2.2) in every localization (cf. Remark 2.1 below). θ will be called the *2-form associated to* $\langle\,\rangle$.

Remark 2.1. Flanders ([4], Th. 3) has proved that if $\Lambda^n P$ is free on one generator over R, then P is a finitely generated projective R-module (and it follows immediately that P has rank n in every localization).

Remark 2.2. Oriented quadratic forms are defined in ([10], Def. 1.8).

Remark 2.3. If $(P, \langle\ \rangle, \epsilon)$ is an object in $\underline{\underline{SO(n, R)}}$ then $\langle \epsilon, \epsilon \rangle$ is a unit in R (since the inner product induced on $\Lambda^n P$ is non-singular).

For the remainder of this section, we shall assume that n is odd:

$$n = 2l + 1$$

DEFINITION 2.2. Let $(P, \langle,\rangle, \epsilon)$ be a non-singular oriented quadratic form of odd rank $n = 2l + 1$; then by the *special orthogonal flag-algebra* of $(P, \langle,\rangle, \epsilon)$ will be meant the R-algebra

$$\Lambda_{SO}^{+}(P, \langle,\rangle, \epsilon) = \Lambda^{+}P/I(P, \langle\ \rangle, \epsilon) \tag{2.3}$$

where $\Lambda^+ P$ is the R-algebra given by ([8], Def. 1.3), while $I(P, \langle,\rangle, \epsilon)$ is the ideal in $\Lambda^+ P$ generated by all

$$*_\epsilon \alpha - \alpha \qquad (\alpha \in \Lambda^p P, 0 \leqslant p \leqslant l)$$

If $\alpha = \langle a_1, ..., a_s \rangle$ is a "partition" i.e. finite unordered sequence of integers (in the sense explained on p. 84 of [8]) with all $a_i \leqslant l$, then we shall denote the image of $\Lambda^\alpha P$ in 2.3) by $\Lambda_{SO}^{\alpha}(P, \langle\ \rangle, \epsilon)$, and an element in the latter module will be called a *special orthogonal shape in* $(P, \langle,\rangle, \epsilon)$ *of degree* α.

Remark 2.4. Denote by $M(l)$ the set of all partitions, all of whose elements are l. If we define addition of partitions by $\langle a_1, ..., a_s \rangle + \langle b_1, ..., b_t \rangle = \langle a_1, ..., a_s, ..., b_1, ..., b_t \rangle$ then $M(l)$ is a nonoid (with neutral element the empty partition) which grades Λ_{SO}^{+}, i.e.,

$$\Lambda_{SO}^{+} = \bigoplus_{\alpha \in M(l)} \Lambda_{SO}^{\alpha}$$

and the product of Λ_{SO}^{α} and Λ_{SO}^{β} is contained in $\Lambda_{SO}^{\alpha+\beta}$.

From this point until the statement of Theorem 2.3, we shall temporarily assume that R is an algebraically closed field of characteristic 0, over which $(P, \langle,\rangle, \epsilon)$ is a fixed non-singular oriented quadratic module of odd rank $n = 2l + 1$, with $\langle \epsilon, \epsilon \rangle = 1$.

Our immediate goal is the proof that the special orthogonal group

$$G = \mathrm{Aut}(P, \langle\ \rangle, \epsilon)$$

has a flag-algebra $\Lambda^+(G)$ (as defined in sect. 1) which is indeed given by the construction of Definition 2.2.

We shall say that a basis

$$\{e_1, ..., e_n\} \tag{2.4}$$

for P over R is *adapted to* $\langle\,\rangle$, if we have

$$\langle e_i\,, e_j\rangle = \begin{cases} 1 & \text{if } \; i+j=n+1 \\ 0 & \text{otherwise;} \end{cases}$$

we shall say (2.4) is *adapted to* ϵ, if we have

$$e_1 \wedge \cdots \wedge e_n = \epsilon$$

Note that if (2.4) is adapted to both $\langle\,\rangle$ and ϵ, then we may compute $*_\epsilon$ by the formula

$$*_\epsilon(e_{h_1} \wedge \cdots \wedge e_{h_p}) = (-1)^\sigma e_{n+1-k_q} \wedge \cdots \wedge e_{n+1-k_1} \tag{2.5}$$

where

$h_1 < \cdots < h_p$; $k_1 < \cdots < k_q$; $\{1,..., n\}$ is the disjoint union of $\{h_1\,,..., h_p\}$ and $\{k_1\,,..., k_q\}$,

and

$$\sigma = \sum_{t=1}^{p} (h_t - t) + (n-p)(n-p-1)/2.$$

Let us denote by $\mathscr{B}$ the subset of

$$nP = P \oplus \cdots \oplus P \,(n \text{ times})$$

consisting of all bases for P adapted to both $\langle\,\rangle$ and ϵ; this is a Zariski-closed non-empty subset of nP, irreducible since G acts transitively on it. Now set

$$\varphi\colon nP \to \Lambda P,\ (e_1\,,..., e_n) \to \sum_{i=1}^{n} e_1 \wedge \cdots \wedge e_i$$

$$\varphi_{1/2} : nP \to \Lambda P,\ (e_1\,,..., e_n) \to \sum_{i=1}^{l} e_1 \wedge \cdots \wedge e_i$$

and observe the isomorphisms of varieties

$$G/U \approx \varphi(\mathscr{B}) \approx \varphi_{1/2}(\mathscr{B}),$$

the latter isomorphism being based on the observation that if $\{e_1\,,...., e_n\} \in \mathscr{B}$, then

$$*(e_1 \wedge \cdots \wedge e_p) = (-1)^{(n-p)(n-p-1)/2}\, e_1 \wedge \cdots \wedge e_{n-p} \tag{2.6}$$

By the *flag variety* Flag(P) of P will be meant the Zariski closure of $\varphi(nP)$ in ΛP (cf. [7], sect. 3). The Zariski-closure of $\varphi(\mathscr{B})$ in ΛP will be denoted by

$$\text{Ad Flag} = \text{Ad Flag}(P, \langle,\rangle, \epsilon),$$

the *adapted flag variety* of $(P, \langle\ \rangle, \epsilon)$; this is a closed irreducible subvariety of Flag(P). Finally, the Zariski-closure of $\varphi_{1/2}(\mathscr{B})$ in ΛP will be denoted by

$$\text{Iso Flag} = \text{Iso Flag}(P, \langle\ \rangle, \epsilon),$$

the *isotropic flag variety* of $(P, \langle\ \rangle, \epsilon)$; this may also be defined as the Zariski-closure in ΛP of the set of all

$$e_1 + (e_1 \wedge e_2) + \cdots + (e_1 \wedge \cdots \wedge e_l)$$

where $e_1, \ldots, e_l$ are isotropic orthogonal elements of P.

We observe that R[Iso Flag], and so the isomorphic algebra R[Ad Flag] is naturally graded by the monoid $M(l)$ (of Remark 2.5) as follows: if

$$\alpha = \langle a_1, \ldots, a_s \rangle,\ a_1 \geqslant \cdots \geqslant a_s > 0$$

then we define

$$b_i = \#\{j\colon a_j \geqslant i\} \qquad (1 \leqslant i \leqslant l)$$

Thus,

$$b_1 \geqslant \cdots \geqslant b_l \geqslant 0$$

and, omitting any 0's among the b's, they make up the partition conjugate to α. We then define

$$R[\text{Iso Flag}]^{\alpha}$$

to consist of those f in R[Iso Flag] such that

$$f(x_1 + x_1 \wedge x_2 + \cdots + x_1 \wedge \cdots \wedge x_l)$$

is homogeneous of degree b_i in the coordinates of x_i for $1 \leqslant i \leqslant l$. (Note that if α, α' are thus associated with $b_1 \geqslant \cdots \geqslant b_l \geqslant 0$ and $b_1' \geqslant \cdots \geqslant b_l' \geqslant 0$ respectively, then $\alpha + \alpha'$ is associated with $b_1 + b_1' \geqslant \cdots \geqslant b_l + b_l' \geqslant 0$ so this does yield a gradation).

We also use these b's to define the graded component $\Lambda^{[\alpha]}(G)$ of $\Lambda^{+}(G)$ (which thus acquires a gradation by $M(l)$) via

$$\Lambda^{[\alpha]}(G) = \Lambda^{\beta} G$$

with β the weight associated, by the usual conventions (cf. Weyl, [12], p. 155) with the sequence $b_1 \geqslant \cdots \geqslant b_l \geqslant 0$. In other words, $f \in k[G/U]$ is in $\Lambda^{\alpha}(G)$ and only if it has the following property:

Let $\bar{e}_1, \ldots, \bar{e}_n$ be an adapted basis which we may assume chosen so all elements of U are represented by upper triangler matrices; for $\lambda_1, \ldots, \lambda_l$ non-zero elements of k, denote by $\Delta(\lambda_1, \ldots, \lambda_l)$ the element of G represented with respect to this basis by $\mathrm{diag}\{\lambda_1, \ldots, \lambda_l, 1, \lambda_l^{-1}, \ldots, \lambda_1^{-1}\}$; then, for all x in U,

$$f(x\Delta(\lambda_1, \ldots, \lambda_l)U) = \lambda_1^{b_1} \cdots \lambda_l^{b_l} f(xU).$$

LEMMA 2.1. *There are isomorphisms*

$$\Lambda^+(G) \approx R[\text{Iso Flag}] \approx R[\text{Ad Flag}] \tag{2.7}$$

of $M(l)$-graded R-algebras.

Caution. This result breaks down in several bizarre ways when $\dim P$ is even. While G/U is a dense open subset of Ad Flag, $R[G/U]$ is not isomorphic to R[Ad Flag] in the even-dimensional case; also, while

$$\Lambda^+(G) = R[G/U]$$

remains isomorphic to R[Iso Flag] as a graded R-module, they are not then isomorphic as R-algebras, i.e., the multiplication in the coordinate ring of Iso Flag is not always the Cartan product (cf. [11] for a further discussion of these matters).

Proof of Lemma 2.1. The second isomorphism is immediate. In order to prove the first isomorphism, we begin by picking any adapted basis $\{e_1, \ldots, e_n\}$ in $\mathscr{B}$, and selecting accordingly the associated Borel subgroup B of G, consisting of all elements of G represented with respect to this basis by upper triangular matrices. The map

$$G/U \to \text{Iso Flag}, gU \mapsto \varphi_{1/2}(ge_1, \ldots, ge_n)$$

exhibits G/U as a dense open subset of Iso Flag, whence the $M(l)$-graded R-algebra-monomorphism

$$\iota\colon R[\text{Iso Flag}] \to \Lambda^+(G)$$

It suffices to show that the image of ι contains all fundamental representations

$$\Lambda^{[\langle i\rangle]}(G) \approx \Lambda^i P \qquad (1 \leqslant i \leqslant l)$$

To see this, it suffices to show that $R[\text{Ad Flag}]^{\langle i\rangle}$ contains a submodule isomorphic to $\Lambda^i P$ (whose image by the monomorphism ι is therefore all of $\Lambda^{[\langle i\rangle]}(G)$). For this, it suffices to note that the G-equivariant map

$$\kappa\colon \Lambda^i P \to R[\text{Iso Flag}]$$

defined by

$$(\kappa\omega)(\omega_1 + \cdots + \omega_l) = \langle \omega, \omega_i \rangle$$

(where $\omega \in \Lambda^i P$, $\omega_1 + \cdots + \omega_l \in \text{Iso Flag}$ with $\omega_p \in \Lambda^p P$ for $1 \leqslant p \leqslant l$) is not identically 0; e.g.

$$(\kappa e_{n-i+1} \wedge \cdots \wedge e_{n-1} \wedge e_n)(e_1 + \cdots + (e_1 \wedge \cdots \wedge e_n)) = \pm 1.$$

Q.E.D.

THEOREM 2.2. *Let $(P, \langle,\rangle, \epsilon)$ be a non-singular oriented quadratic module of odd rank $2n + 1$ over an algebraically closed field R of characteristic 0, and let*

$$G = \mathrm{Aut}(P, \langle,\rangle, \epsilon)$$

be the associated special orthogonal group.

Then the flag-algebra $\Lambda^+(G)$ of G (as discussed in sect. 1) is isomorphic to $\Lambda^+_{SO}(P, \langle\ \rangle, \epsilon)$, and every finite-dimensional irreducible polynomial representation of G is equivalent to one of the form

$$G \to \mathrm{Aut}(\Lambda^\alpha_{SO}(P, \langle\ \rangle, \epsilon)),\ T \to \Lambda^\alpha_{SO} T$$

for a unique partition α, all of whose elements are $\leqslant l$.

Remark 2.5. The irreducible representation of $SO(2l + 1)$ associated with Λ^α_{SO} is that associated (in the usual notation) to the sequence $b_1 \geqslant \cdots \geqslant b_l \geqslant 0$ which gives the partition conjugate to α.

Proof. Let us retain the notation of the preceding discussion; thus, it suffices to prove the existence of a natural isomorphism

$$\Lambda^+_{so}(P, \langle\ \rangle, \epsilon) \approx R[\text{Ad Flag}(P, \langle\ \rangle, \epsilon)] \tag{2.8}$$

Since

$$\text{Ad Flag} \subset \text{Flag}(P),\ R[\text{Flag } P] \approx \Lambda^+ P^*$$

we have the composite R-algebra epimorphism

$$\psi\colon \Lambda^+ P \xrightarrow[\Lambda^+\iota]{\approx} \Lambda^+ P^* \xrightarrow[\text{epic}]{} R[\text{Ad Flag}]$$

(where $\iota = \iota_{\langle\rangle}$ is the isomorphism of Def. 2.1).

Claim. Ker ψ *is the ideal I' in $\Lambda^+ E$ generated by all*:

$$*\alpha - (-1)^{p(p-1)/2}\alpha \qquad (\alpha \in \Lambda^p P, 0 \leqslant p \leqslant l) \tag{2.9}$$

Before verifying this, let us note the existence of a unique natural isomorphism $\theta\colon \Lambda^+ P \to \Lambda^+ P$, which is the identity on $\Lambda^p P$ for $l < p \leqslant n$, and is multiplication by $(-1)^{p(p-1)/2}$ on $\Lambda^p P$ for $0 \leqslant p \leqslant l$; then the epimorphism

$$\psi \circ \theta\colon \Lambda^+ P \to R[\text{Ad Flag}]$$

has as kernel the ideal $I(P, \langle\ \rangle, \epsilon)$ of Definition 2.2. Thus, the proof of the present theorem will be complete once the preceding claim is verified.

It is easy to see that Ker ψ contains all elements of the form 2.9; indeed, if $\phi e_1, ..., e_n\} \in \mathscr{B}$ is an adapted basis, the equality of $*\alpha$ and $(-1)^{p(p-1)/2}\alpha$ on $\{(e_1, ..., e_n)$ is a straightforward consequence of (2.5).

The difficult part is to show that

$$I' \supseteq \operatorname{Ker} \psi$$

We begin by noting that (by a straightforward computation involving 2.5) I' contains all

$$*\alpha - (-1)^{p(p-1)/2}\alpha \qquad (\alpha \in \Lambda^p P, l < p \leqslant n). \tag{2.10}$$

To complete the proof, it suffices to verify that the composite epimorphism

$$\psi_1 : \Lambda^+ P \xrightarrow{\psi} R[\text{Ad Flag}] \xrightarrow{\approx} \Lambda^+(G)$$

has kernel contained in I'. Note that, for $1 \leqslant i \leqslant l$,

$$\psi_1(\Lambda^i P) \neq 0$$

as may be seen from

$$\psi(e_{n-i+1} \wedge \cdots \wedge e_n)(e_1 + \cdots + e_1 \wedge \cdots \wedge e_n) = \pm 1$$

and since the G-modules P, $\Lambda^2 P, ..., \Lambda^l P$ give the fundamental weights W_0 in Theorem 1, $\pi_{\alpha_1}, ..., \pi_{\alpha_l}$ of G, they are therefore mapped isomorphically by ψ_1 onto the components $\Lambda^{\alpha_1}(G), ..., \Lambda^{\alpha_l}(G)$ of $\Lambda^+(G)$. Let T denote the tensor algebra on

$$\bigoplus_{i=1}^{l} \Lambda^{\alpha_i} G$$

and let KR denote the ideal in T generated by the Kostant relations (1.2); then there is an exact sequence

$$0 \to KR \to T \underset{\phi}{\to} \Lambda^+(G) \to 0$$

with ϕ factoring as

$$T \xrightarrow{\alpha} \Lambda^+ P \xrightarrow{\Psi_1} \Lambda^+(G)$$

and with $\Lambda^+ P = 1m\alpha + I'$. Thus, the proof will be complete if we can show that I' contains $\alpha(KR)$.

If we pick as basis for the Lie algebra of G (with respect to an adapted basis)

$$X_{rs} = E^{r,s} - E^{n+1-s,n+1-r}$$

($r + s < n + 1$, $E^{r,s}$ has 1 in (r, s) entry and 0 elsewhere) with dual basis $X^*_{rs} = \frac{1}{2}X_{sr}$ then a straightforward computation shows the Kostant relations (1.1) are the images under ψ_1 of expressions in Λ^+E of the following kind:

$$P(e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes e_{j_1} \wedge \cdots \wedge e_{j_q})$$
$$= \sum_{\substack{\lambda=1 \\ i_\lambda + j_\mu \neq n+1}}^{p} \sum_{\mu=1}^{q} T_{\lambda\mu} - q(e_{i_1} \wedge \cdots \wedge e_{i_p}) \cdot (e_{j_1} \wedge \cdots \wedge e_{j_q}) - \sum_{\lambda=1}^{p} \sum_{\mu=1}^{q} \sum_{\substack{r=1 \\ r \neq n+1-i_\lambda}}^{n} T'_{\lambda\mu r}$$

where $l \geqslant p \geqslant q$, $T_{\lambda\mu}$ is the result of interchanging e_{i_λ} and e_{j_μ} in

$$(e_{i_1} \wedge \cdots \wedge e_{i_p}) \cdot (e_{j_1} \wedge \cdots \wedge e_{j_q}),$$

and $T'_{\lambda\mu r}$ is the product of $\langle e_{i_\lambda}, e_{i_\mu}\rangle$ by the result of replacing e_i by e_r and e_{j_μ} by e_{n+1-r} in $(e_{i_1} \wedge \cdots \wedge e_{i_p}) \cdot (e_{j_1} \wedge \cdots \wedge e_{j_q})$.

Utilizing the Young symmetry relations in Λ^+P (cf. [8], Def. 1.3), it is immediate that

$$\sum_{\lambda=1}^{p} \sum_{\mu=1}^{q} T_{\lambda\mu} = q(e_1 \wedge \cdots \wedge e_p) \cdot (e_{j_1} \wedge \cdots \wedge e_q)$$

whence (2.10) simplifies to

$$P(e_1 \wedge \cdots \wedge e_p \otimes e_{j_1} \wedge \cdots \wedge e_{j_q})$$
$$= - \sum_{i_\lambda + j_\mu = n+1} Q(e_1 \wedge \cdots \wedge \hat{e}_{i_\lambda} \wedge \cdots \wedge e_{i_p} \otimes e_{j_1} \wedge \cdots \wedge \hat{e}_{j_\mu} \wedge \cdots \wedge e_{j_q})$$

where, for $l > p \geqslant q$,

$$Q(e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes e_{j_1} \otimes \cdots \otimes e_{j_q})$$
$$\sum_{\lambda+\mu=n+1} (e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_\lambda) \cdot (e_{j_1} \wedge \cdots \wedge e_{j_q} \wedge e_\mu) \tag{2.11}$$

Thus, we are done if we show that I' contains all expressions of the form (2.11). We begin by noting that, modulo I', the term

$$(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_1) \cdot (e_{j_1} \wedge \cdots \wedge e_{j_q} \wedge e_n)$$

is congruent to

$$\pm [*(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_1)] \cdot (e_{j_1} \wedge \cdots \wedge e_{j_q} \wedge e_n]$$

If we apply the Young symmetry rule to "switch" e_n from the second term to the first, and then apply $*$ again to the first factor of each term in the resulting sum, we will obtain (modulo I') the result

$$- \sum_{\lambda=2}^{n} (e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_\lambda)(e_{j_1} \wedge \cdots \wedge e_{j_q} \wedge e_{n+1-\lambda})$$

whence (as was to be proved), I' contains the expression (2.11). This completes the proof of Theorem 2.2.

From this point we drop the assumptions: R is a field of characteristic 0, $\langle \epsilon, \epsilon \rangle = 1$, $\{e_1, \dots, e_n\}$ is adapted.

THEOREM 2.3. *Let R be a field of characteristic $\neq 2$, let $(P, \langle\ \rangle, \epsilon)$ be a quadratic R-module of odd rank $n = 2l + 1$, and assume P is free over R on $\{e_1, \dots, e_n\}$; denote the dual basis by $\{e_1^*, \dots, e_n^*\}$.*

Then $\Lambda_{so}^+(P, \langle\ \rangle)$ is generated as an R-algebra by

$$\bigotimes_{i=1}^{l} \Lambda^i P \tag{2.12}$$

with relations generated by all

$$0 = \sum_{i_1 < \cdots < i_h} \sum_{j_1, < \cdots, < j_h} (\alpha \wedge e_{i_1} \wedge \cdots \wedge e_{i_h})(\beta \wedge e_{j_1} \wedge \cdots \wedge e_{j_h}) \tag{2.13}$$

$$\times \langle e_{i_1}^* \wedge \cdots \wedge e_{i_h}^*, e_{j_1} \wedge \cdots \wedge e_{j_h} \rangle \ (\alpha \in \Lambda^p P, \beta \in \Lambda^q P, p + h \text{ and } q + h \leqslant l)$$

If R is a $\mathbb{Q}$-algebra, it suffices to consider only such relations with $h = 1$.

Proof. It is readily verified that if (2.13) holds, and if we modify the basis $\{e_1, \dots, e_n\}$ by replacing R_i by ce_i $(c \neq 0)$ or by $e_i + ce_i$ $(i \neq j)$, then (2.13) remains valid for the resulting basis. Hence (2.13) holds for all R-bases for P if it holds for one, and we may assume without loss of generality that

$$\langle e_i, e_j \rangle = \delta_j^i \cdot a_i \qquad (a\text{'s} \neq 0)$$

Denote by c the non-zero element of R such that

$$e_1 \wedge \cdots \wedge e_n = c\epsilon \qquad (\text{whence } a_1 \cdots a_n = c^2 \langle \epsilon, \epsilon \rangle)$$

Under these hypotheses, we may compute $*_\epsilon$ by:

$$\begin{aligned} &*_\epsilon e_{i_1} \wedge \cdots \wedge e_{i_h} \\ &= c \cdot sgn \begin{pmatrix} 1 \cdots n \\ i_1 \cdots i_h j_1 \cdots j_{n-h} \end{pmatrix} \cdot (e_{j_1} \wedge \cdots \wedge e_{j_{n-h}}) \cdot \frac{1}{\prod_{\lambda=1}^{R} a_{i_\lambda}} \end{aligned} \tag{2.14}$$

where $i_1 < \cdots < i_h$, $j_1 < \cdots < j_{n-h}$, and $\{1, \dots, n\}$ is the disjoint union of $\{i_1 \cdots i_h\}$ and $\{j_1, \dots, j_{n-h}\}$.

We may obtain (2.13) from the generating relations

$$\omega = *\omega \qquad (\omega \in \Lambda^p P, p \leqslant l)$$

together with their consequences (2.10), by the following process (which was already used in the preceding proof in connection with the expressions (2.11) and "adapted bases"):

Let $\bar{i}_1 < \cdots < \bar{i}_h$ be the h *smallest* elements i such that e_i does not occur in α, and form the expression

$$(\alpha \wedge e_{\bar{i}_1} \wedge \cdots \wedge e_{\bar{i}_h})(\beta \wedge e_{\bar{i}_1} \wedge \cdots \wedge e_{\bar{i}_h})$$

Apply $*$ to the first term, use Young symmetry to switch $e_{\bar{i}_1} \wedge \cdots \wedge e_{\bar{i}_h}$ from the second term into the resulting first term, and in each summand of the result, apply $*$ again to the first term. The result is, that

$$(\alpha \wedge e_{\bar{i}_1} \wedge \cdots \wedge e_{\bar{i}_h})(\beta \wedge e_{\bar{i}_1} \wedge \cdots \wedge e_{\bar{i}_h}) \cdot \frac{1}{a_{\bar{i}_1} \cdots a_{\bar{i}_h}}$$

is expressed as the sum of the other terms in (2.13).

Conversely, we must show the equations (2.13) generate the relations on (2.12). Denote by $\Lambda^+_{1/2}P$ the sub-algebra of Λ^+P generated by (2.12), and by I the ideal generated in $\Lambda^+_{1/2}P$ by the relations (2.13). Denote by ψ_0 the natural R-epimorphism

$$\psi_0 : S\Lambda P \to \Lambda^+_{1/2}P$$

$$\text{which maps } \alpha \text{ in } \Lambda^p P \text{ into } \begin{cases} \alpha & \text{if } \; p \leqslant l, \\ (-1)^l * \alpha & \text{if } \; p > l \end{cases}$$

and by ψ the composite epimorphism

$$\psi\colon S\Lambda P \xrightarrow{\psi_0} \Lambda^+_{1/2}P \to \Lambda^+_{1/2}\, P/\tilde{I}$$

It suffices to verify that Ker ψ coincides with the ideal I' in $S\Lambda P$ generated by the Young symmetry relations used to define Λ^+P, together with the generators $\alpha - *\alpha$ ($\alpha \in \Lambda^pP$, $p \leqslant l$) of the ideal $I(P, \langle\,\rangle, \epsilon)$ of Definition 2.2. It is clear I' contains the later elements. Next, consider the Young symmetry relations

$$\omega - \sum_{\lambda=1}^{p} ((e_{i_\lambda}, e_{j_1})) \circ \omega \tag{2.15}$$

(cf. [8], p. 83), where

$$\omega = (e_{i_1} \wedge \cdots \wedge e_{i_p}) \otimes (e_{j_1} \wedge \cdots \wedge e_{j_q}),\; p \geqslant q.$$

If $l \geqslant p$, these lie in I' because ψ factors through $\Lambda^+_{1/2}P$. If $p > l \geqslant q$, we may obtain these by "starring" the first term of each summand of (2.13), as in the earlier part of the proof. If $p \geqslant q > l$, we obtain (2.14) in I' by "starring" both parts of a Young symmetry relation between terms of lengths $n - p$ and $n - q$. If char $R = 0$, the relations (2.15) generate all Young relations, so, as asserted, we need only use relations (2.13) with $h = 1$. If char $R \neq 0$, we use the "higher" relations (2.13) (with $h > 1$) to obtain the "higher" Young relations, in the same way.

This shows that Ker ψ contains I'; since we also know that Ker ψ is generated by $\tilde{I}$ and the elements $\alpha - *\alpha$ ($\alpha \in \Lambda^pP, p \leqslant l$) in I', and since the first part of the proof shows that $\tilde{I} \subseteq I'$, we may conclude that $\operatorname{Ker} \psi = I'$, as was to be proved.

THEOREM 2.4. *Let R be a field of characteristic* 0, *and let* $(P, \langle\ \rangle)$ *be a non-singular orientable quadratic R-module of odd rank $n = 2l + 1$; let*

$$\Theta \in D_2 P = S^2 P$$

be the associated quadratic form. Let $((\Theta))$ denote the "double ideal" of $\Lambda^+ P$ generated by Θ, i.e., the smallest subset of $\Lambda^+ P$ which contains Θ and is closed under R-linear combinations, the usual commutative product in $\Lambda^+ P$, and the "extra" wedge product in $\Lambda^+ P$ (as defined in [8], p. 101).

Then if we denote by $\Lambda^+_{1/2} P$, the sub-algebra of $\Lambda^+ P$ (with respect to the original commutative multiplication$_s$) generated by

$$\bigotimes_{i=1}^{l} \Lambda^i P,$$

there is a natural isomorphism

$$\Lambda^+_{so}(P, \langle\ \rangle) \approx \Lambda^+_{1/2} P / [\Lambda^+_{1/2} P \cap ((\theta))].$$

Proof. In the first place, it is not difficulty to verify that the "double ideal" $((\Theta))$ generated by Θ, coincides with the ideal (in the usual sense) in the ring $\Lambda^+ P$ (with the original commutative multiplication) generated by Θ and by all $\alpha \wedge \Theta$ (where $\alpha \in \Lambda^p P$ or $\Lambda^{p,q} P$, p and q between 1 and $n - 1$).

Now, for any partitions α and β, the "extra" wedge product in $\Lambda^+ P$, restricted to $\Lambda^\alpha P \otimes \Lambda^\beta P$, maps into $\Lambda^{(\alpha^* + \beta^*)^*}$ (where α^* denotes the partition conjugate to α, and the operation $+$ is as defined on p. 85 of [8]); moreover, this is the unique natural transformation

$$\Lambda^\alpha \otimes \Lambda^\beta \to \Lambda^{(\alpha^* + \beta^*)^*}$$

(to within scalar multiples) since the multiplicity with which the representation of $GL(n)$ associated with $(\alpha^* + \beta^*)^*$ occurs in the tensor product of those associated with α and with β respectively, is 1.

If $\alpha = \langle p, q \rangle$, with $n > p \geqslant q$, and $\beta = \langle 1, 1 \rangle$, then $(\alpha^* + \beta^*)^* = \langle p + 1, q + 1 \rangle$, and the map

$$\begin{gathered} \Lambda^{p,q} P \otimes \Lambda^{1,1} P \to \Lambda^{p+1,q+1} P, \\ (\alpha \cdot \beta) \otimes (z_1 \cdot z_2) \mapsto (\alpha \wedge z_1) \cdot (\beta \wedge z_2) + (\alpha \wedge z_2) \cdot (\beta \wedge z_1) \qquad (2.16) \\ (\alpha \in \Lambda^p P, \beta \in \Lambda^q P, z_1 \text{ and } z_2 \text{ in } P) \end{gathered}$$

must coincide with the wedge product (up to non-zero rational multiples) provided it is well-defined and non-zero.

(2.16) is well-defined, since

$$\begin{aligned} T_1 + T_2 = {} & (x_1 \wedge \cdots \wedge x_p \wedge z_1) \cdot (y_1 \wedge \cdots \wedge y_q \wedge z_2) \\ & + (x_1 \wedge \cdots \wedge x_p \wedge z_2) \cdot (y_1 \wedge \cdots \wedge y_q \wedge z_1) \end{aligned}$$

goes into itself if we replace y_1 by $x_1, \ldots, x_p$ in turn, and all the p resulting expressions, indeed, the result, differs from

$$T_1 + T_2 = \sum_{i=1}^{p} ((x_i, y_1))(T_1 + T_2) + ((y_1, z_1))\, T_1 + ((y_1, z_2))T_2$$

by the expression

$$0 = (x_1 \wedge \cdots \wedge x_p \wedge y_1) \cdot (z_1 \wedge y_2 \wedge \cdots \wedge y_q \wedge z_2 + z_2 \wedge y_2 \wedge \cdots \wedge y_q \wedge z_1).$$

To see that (2.16) is not identically 0, when $n \geqslant p \geqslant q \geqslant 0$, we observe that if $\{e_1 < \cdots < e_n\}$ is an ordered basis for P, then

$$[(e_1 \wedge \cdots \wedge e_p) \cdot (e_1 \wedge \cdots \wedge e_q)] \wedge e_{p+1}^2 = 2(e_1 \wedge \cdots \wedge e_p \wedge e_{p+1})(e_1 \wedge \cdots \wedge e_q \wedge e_{p+1}$$

is "standard" with respect to the given basis (in the sense of [8], Def. 2.1) and hence non-zero by [8], Theorem 2.4). [Alternatively, it suffices to consider the case when R is a field of characteristic 0; here Λ^+P is a domain, since it is the coordinate ring of an irreducible variety Flag (P^*).]

Since it has been established that (2.16) computes $\wedge$ (in the relevant special case), it follows that

$$\begin{aligned}&[(x_1 \wedge \cdots \wedge x_p)(y_1 \wedge \cdots \wedge y_q)] \wedge \theta \\ &\quad = \sum_{i+j=n+1} (x_1 \wedge \cdots \wedge x_p \wedge e_i)(y_1 \wedge \cdots \wedge y_q \wedge e_j) \langle e_i^*, e_j^* \rangle,\end{aligned}$$

whence the ideal $\Theta \cap \Lambda_{1/2}^+(P)$ in $\Lambda_{1/2}^+(P)$ coincides with those relations (2.12) for which $h = 1$; the present theorem is thus a consequence of Theorem 2.3.

3. The Symplectic Groups

As is very often the case, so here also the groups

$$B_l = SO(2l + 1) \qquad \text{and} \qquad C_l = Sp(l)$$

exhibit quite similar behavior. We shall now state constructions and related theorems for $Sp(l)$ which are analogs of the results in Section 2; as for the proofs, we shall content ourselves with only sketching those few places where they are not simply duplicates of the previous proofs.

R continues to denote a commutative associative ring with 1.

Definition 3.1. By a *non-singular symplectic R-module of rank* $n = 2l$ will be meant an ordered pair $(P, \langle\ \rangle)$ *where* P is an R-module and

$$\langle, \rangle : P \otimes_R P \rightarrow R$$

is an alternating R-bilinear form on R, these being subject to the two conditions:

(i) P is a finitely generated R-module, of rank n in every localization.

(ii) The R-homomorphism

$$\iota_{\langle\,\rangle}: E \to E^*, \qquad e \mapsto \langle e, -\rangle$$

induced by $\langle , \rangle$, is an isomorphism.

These form a category $\underline{\underline{Sp(l, R)}}$, whose morphisms are those R-isomorphisms which preserve $\langle , \rangle$.

If $(P, \langle , \rangle)$ is as above, and if P is R-free, then there exists an R-basis $\{e_1, \ldots, e_n\}$ for P such that

$$\langle e_i, e_j\rangle = \begin{cases} 1 \text{ if } i + j = n + 1 \text{ and } i < j; \\ 0 \text{ if } i + j \neq n + 1 \end{cases} \tag{3.1}$$

and such a basis will be called *adapted to* $\langle\,\rangle$; we then set

$$\omega = \omega(P, \langle\,\rangle) = \sum_{i+j=n+1} e_i \wedge e_j \in \Lambda^2 P \tag{3.2}$$

(the "associated 2-form"), and

$$\epsilon = \epsilon(P, \langle\,\rangle) = e_1 \wedge \cdots \wedge e_n \in \Lambda^n E \tag{3.3}$$

(the "associated orientation"); these are independent of the choice of adapted basis.

If P is not free, ω and ϵ are still uniquely specified by the requirement that they be given by (3.2) and (3.3) in every localization.

For $1 \leqslant p \leqslant n$, we denote by $\langle\,\rangle_p$ the R-bilinear form (symmetric or symplectic, according as p is even or odd) defined by

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p\rangle_p = \det(\langle v_i, w_j\rangle)_{1\leqslant i\leqslant p, 1\leqslant j\leqslant p},$$

while the R-isomorphism

$$* = *_p = *_p(P, \langle\,\rangle): \Lambda^p P \xrightarrow{\approx} \Lambda^{n-p} P \tag{3.4}$$

is uniquely determined by the requirement

$$\alpha \wedge \beta = \langle *\alpha, \beta\rangle_{n-p}\epsilon \tag{3.5}$$

for all $\alpha \in \Lambda^p P$, $\beta \in \Lambda^{n-p} P$.

Remark 3.1. If $\{e_1, \ldots, e_n\}$ is an adapted basis, then $*$ may be computed explicitly by the formula

$$*(e_{h_1} \wedge \cdots \wedge e_{h_p}) = (-1)^{\sigma}\, e_{n+1-k_q} \wedge \cdots \wedge e_{n+1-k_1} \tag{3.6}$$

where: $h_1 < \cdots < h_p$, $k_1 < \cdots < k_q$; $\{1,\ldots, n\}$ is the disjoint union of $\{h_1, \ldots, h_p\}$ and $\{k_1, \ldots, k_q\}$;

$$\sigma = \sum_{t=1}^{p} (h^q - t) + (n-p)(n-p-1)/2 + \#\{i : k_i \leqslant l)$$

Remark 3.2. This definition implies (by the construction for ϵ) that $\Lambda^n P$ is free of rank 1, as in Definition 2.1.

DEFINITION 3.2. Let $(P, \langle, \rangle)$ be a non-singular symplectic R-module of rank $n = 2l$; then by the *symplectic flag-algebra* of $(P, \langle\ \rangle)$ will be meant the R-algebra

$$\Lambda_S{}^+(P, \langle\ \rangle) = \Lambda^+ P/I(P, \langle\ \rangle) \tag{3.7}$$

where $I(P, \langle\ \rangle)$ is the ideal in $\Lambda^+ E$ generated by all

$$\alpha \wedge \omega \qquad (\alpha \in \Lambda^p P, 0 \leqslant p \leqslant l-2) \tag{3.8}$$

together with all

$$\alpha - *\alpha \qquad (\alpha \in \Lambda^p P, 0 \leqslant p < l) \tag{3.9}$$

(with ω, $*$ given by Def. 2.1).

If $\alpha = \langle a_1, \ldots, a_s\rangle$ is a partition, with all $a_i \leqslant l$, then we shall denote the image of $\Lambda^p P$ in (3.7) by $\Lambda_S{}^\alpha(P, \langle\ \rangle)$, and the image of $\prod_{i=1}^{s}(x_{i_1} \wedge \cdots \wedge x_{i,h_i})$ by $\prod_{i=1}^{s}\{x_{i_1} \wedge \cdots \wedge x_{i,h_i}\}$; the latter will be called a *symplectic shape of degree* α *in* $(P, \langle, \rangle)$.

Remark 3.3. The ideal generated by the elements (3.8) already contains all elements of the form

$$\alpha - (-1)^{l(l-1)/2} * \alpha \qquad (\alpha \in \Lambda^l P)$$

THEOREM 3.1. *Let $P(P, \langle, \rangle)$ be a non-singular symplectic module over an algebraically closed field of characteristic* 0, *and let*

$$G = \mathrm{Aut}(P, \langle\ \rangle)$$

be the associated symplectic group; then the flag-algebra $\Lambda^+(G)$ (as discussed in sect. 1) *is isomorphic to $\Lambda_S{}^+(P, \langle, \rangle)$, and every finite-dimensional irreducible polynomial representation of G is equivalent to one of the form*

$$\mathrm{Aut}(P, \langle\ \rangle) \mapsto \mathrm{Aut}(\Lambda_s{}^\alpha(P, \langle\ \rangle)),\ T \mapsto \Lambda_S{}^\alpha T$$

for a unique partition α, all of whose elements are $\leqslant l$.

Proof. The proof of Theorem 2.2 works, with the following modifications: The fundamental representations of G are here not $\{\Lambda^p P\}$ (as was the case in sect. 2) but rather are those given by the representation modules

$$\Lambda_S{}^p(P, \langle , \rangle) = \Lambda^p P/(\omega \wedge \Lambda^{p-2}P)$$

(with ω given by Definition 2.1, and setting $\Lambda^{-2}P = \Lambda^{-1}P = 0$, $\Lambda^q P = R$). (This may be seen, either by a use of the Casimir operator analogous to that in Theorem 1.1, or by noting that $\Lambda^p P/(\omega \wedge \Lambda^{p-2}P)$ is a G-module whose highest weight is that of one of the fundamental representations, and of dimension $\geqslant$ $\dim \Lambda^p P - \dim \Lambda^{p-2}P =$ dim of that fundamental representation, whence

$$0 \longrightarrow \Lambda^{p-2}P \xrightarrow{\wedge\omega} \Lambda^p P \longrightarrow \Lambda_S{}^p(P, \langle\ \rangle) \longrightarrow 0$$

is exact.)

The definitions of Ad Flag, Iso Flag occur precisely as before, and Lemma 2.1 (and its proof) go over:

$$\Lambda^+(G) \approx R[\text{Ad Flag}] \approx R[\text{Iso Flag}]$$

with no modifications needed, except that in place of (2.6), we have instead, for any adapted basis $\{e_1, ..., e_n\}$,

$$*(e_1 \wedge \cdots \wedge e_p) = \gamma(l, p)\, e_1 \wedge \cdots \wedge e_{n-p} \tag{3.10}$$

with

$$\gamma(l, p) = (n-p)(n-p-1)/2 + \max(l-p, 0)$$

As before, we then set up an R-algebra epimorphism

$$\psi\colon \Lambda^+P \xrightarrow{\approx} \Lambda^+P^* \xrightarrow[\text{epic}]{} R[\text{Ad Flag}]$$

and it suffices to verify that Ker ψ is the ideal I' in Λ^+P generated by all

$$\alpha - (-1)^{\beta(l,i)} * \alpha \qquad (\alpha \in \Lambda^i E, 0 \leqslant i \leqslant n) \tag{3.11}$$

with $\beta(l, i) = i(i-1)/2 + \max(i-l, 0)$, together with all elements of the form (3.8).

In showing that Ker ψ indeed contains these generators for I', the proof for the elements (3.11) is as before, while as for the elements (3.8), we note that if $\{e_1, ..., e_n\}$ is any adapted basis, then

$$\omega = \sum_{i=1}^{n} e_i \wedge e_{n+1-i}$$

so that if $p + 2 \leqslant l$ then

$$\psi(x_1 \wedge \cdots \wedge x_p \wedge \omega)(e_1 + e_1 \wedge e_2 + \cdots + e_1 \wedge e_2 \wedge \cdots \wedge e_n)$$

$$= \sum_{i=1}^{l} \begin{vmatrix} \langle x_1 , e_1\rangle & \cdots & \langle x_1 , e_{p+2}\rangle \\ \vdots & \vdots & \vdots \\ \langle x_p , e_1\rangle & \cdots & \langle x_p , e_{p+2} \\ \langle e_i , e_1\rangle & \cdots & \langle e_i , e_{p+2}\rangle \\ \langle e_{n+1-i} , e_1\rangle & \cdots & \langle e_{n+1-i} , e_{p+2}\rangle \end{vmatrix} \tag{3.12}$$

and since $\langle e_i , e_j\rangle = 0$ unless $i + j = n + 1$, the next-to-the last row of each of these determinants consists entirely of zeroes, whence (3.12) equals 0, i.e., Ker ψ contains $x_1 \wedge \cdots \wedge x_p \wedge \omega$, as asserted.

Finally, the application of Theorem 1.1 goes through as before, with the following modifications: we now have a slightly more complicated basis $\{x_i\}$ for the Lie algebra, consisting of

$$\begin{aligned} X_{rs} &= E^{r,s} - E^{n+1-s,n+1-r} && (r, s \text{ between } 1 \text{ and } l); \\ Y_{r,s} &= E^{r,s} + E^{n+1-s,n+1-r} && (1 \leqslant r \leqslant l-1 \text{ and } l+1 \leqslant s \leqslant n-r \\ & && \text{or } \; 1 \leqslant s \leqslant l-1 \text{ and } l+1 \leqslant r \leqslant n-s); \\ Z_i &= E^{i,n+1-i} && (1 \leqslant i \leqslant n). \end{aligned}$$

while the dual basis $\{X_i^*\}$ with respect to the Killing form consists of $X_{rs}^* = \frac{1}{2}X_{s,r}$, $Y_{rs}^* = \frac{1}{2}Y_{s,r}$, $Z_i^* = Z_{n+1-i}$.

We thus obtain the generating relations

$$\sum_{\lambda=1}^{p}\sum_{\mu=1}^{q} T_{\lambda\mu} - \sum_{\lambda=1}^{p}\sum_{\mu=1}^{q}\sum_{r=1}^{n} \langle e_{i_\lambda} , e_{j_\mu}\rangle\langle e_r , e_{n+1-r}\rangle T_{\lambda,\mu,r} - q\phi = 0$$

where $\phi = \{e_{i_1} \wedge \cdots \wedge e_{i_p}\}\{e_{j_1} \wedge \cdots \wedge e_{j_q}\}$ $(l \geqslant p \geqslant q)$, $T_{\lambda\mu}$ is obtained from ϕ by interchanging e_{i_λ} and e_{j_ν}, and $T_{\lambda,\mu,r}$ is obtained from ϕ by replacing e_{i_λ} with e_r and e_{j_μ} with e_{n+1-r}. As in the earlier situation, this simplifies (using the Young symmetry relations) to the equation

$$\sum_{\lambda+\mu=n+1} \{e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge e_\lambda\} \cdot \{e_{j_1} \wedge \cdots \wedge e_{j_q} \wedge e_\mu\} \langle e_\lambda , e_\mu\rangle = 0$$

which is then verified (as in the proof of Theorem 2.1) to be a consequence of the relations (3.8) and (3.9).

THEOREM 3.2. *Let R be a ring, let $(P, \langle , \rangle)$ be a non-singular symplectic R-module of rank $n = 2l$, free over R on the adapted basis $\{e_1 ,..., e_n\}$.*

Then $\Lambda_S{}^+(P, \langle\, \rangle)$ is generated as an R-algebra by

$$\bigotimes_{i=1}^{l} [\Lambda^i P/\omega \wedge \Lambda^{i-2}P] \tag{3.13}$$

(where ω is the associated 2-form), with relations generated by all

$$\sum_{\substack{i_1<\cdots<i_h \\ j_1<\cdots<j_h}} (\alpha \wedge e_{i_1} \wedge \cdots \wedge e_{i_h})(\beta \wedge e_{j_1} \wedge \cdots \wedge e_{j_h}) \langle e_{i_1} \wedge \cdots \wedge e_{i_h}, e_{j_1} \wedge \cdots \wedge e_{j_h} \rangle \tag{3.14}$$

($\alpha \in \Lambda^p P$, $\beta \in \Lambda^q P$, $p + h$ and $q + h \leqslant l$).

If R is a Q-algebra, it suffices to consider only such relations with $h = 1$.

Proof. Same as that of Theorem 2.3, with this modification: (2.14) is to be replaced by (3.6).

Remark 3.4. De Concini informs us that he has also proved, by quite different methods, that the equations (3.14) generate over fields of any characteristic the prime ideal of the isotropic flag variety for the symplectic group. He informs us that his proof is based on an analog, for the flag variety of the symplectic group, of the Hodge–Young standard basis, and that this new kind of standard basis was in part suggested by recent work of Seshadri, Musili and Lakshmi Bai. While we have not yet seen his proofs, it is our feeling that if such a standard basis exists, it is highly probable that it will play a very important role in any future study of the functors constructed here.

THEOREM 3.3. *Let R be a Q-algebra, and let $P(\langle\ \rangle)$ be a non-singular symplectic R-module of rank $n = 2l$, with associated 2-form ω. Let $((\omega))$ denote the "double ideal" of $\Lambda^+ P$ generated by ω(in the sense explained in Theorem 2.4).*

Then there is a natural isomorphism

$$\Lambda_s{}^+(P, \langle, \rangle) \approx \Lambda^+ P/((\omega)).$$

Proof. Same as that of Theorem 2.4, with these two modifications:

(2.16) is replaced by

$$\begin{aligned}
[(x_1 \wedge \cdots \wedge x_p) \cdot (y_1 \wedge \cdots \wedge y_q)] \wedge z_1 \wedge z_2 \\
= (x_1 \wedge \cdots \wedge x_p \wedge z_1) \cdot (y_1 \wedge \cdots \wedge y_q \wedge z_2) \\
- (x_1 \wedge \cdots \wedge x_p \wedge z_2) \cdot (y_1 \wedge \cdots \wedge y_q \wedge z_1),
\end{aligned}$$

and it must be verified that $\Lambda^{l+1} P$ (hence all $\Lambda^p P$ with $p > l$) is contained in $\omega \wedge \Lambda P$.

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